arXiv:hep-th/9606157v4 1 Oct 1996

Vacuum Structures of Supersymmetric Yang-Mills Theories in $1+1$ Dimensions

Hodaka Oda,[*]   Norisuke Sakai[†]   and   Tadakatsu Sakai[‡]

Department of Physics, Tokyo Institute of Technology

Oh-okayama, Meguro, Tokyo 152, Japan

Abstract

Vacuum structures of supersymmetric (SUSY) Yang-Mills theories in $1+1$ dimensions are studied with the spatial direction compactified. SUSY allows only periodic boundary conditions for both fermions and bosons. By using the Born-Oppenheimer approximation for the weak coupling limit, we find that the vacuum energy vanishes, and hence the SUSY is unbroken. Other boundary conditions are also studied, especially the antiperiodic boundary condition for fermions which is related to the system in finite temperatures. In that case we find for gaugino bilinears a nonvanishing vacuum condensation which indicates instanton contributions.

[*] e-mail: hoda@th.phys.titech.ac.jp
[†] e-mail: nsakai@th.phys.titech.ac.jp
[‡] e-mail: tsakai@th.phys.titech.ac.jp

1. Introduction

Supersymmetric (SUSY) theories offer promising models for the unified theory, but nonperturbative methods are acutely needed to make further progress in understanding such issues as supersymmetry breaking. Recently, good progress has been made on nonperturbative aspects of supersymmetric Yang-Mills theories. Using holomorphy and duality, exact results of the low energy physics of $N = 2$ super Yang-Mills theories were obtained [1]. As for $N = 1$ super Yang-Mills theories, further insight has been gained within the context of duality [2]. These results deal with low energy effective theories, and are based on general but rather indirect arguments. It is perhaps illuminating to study the supersymmetric gauge theories by more dynamical calculations. Since it is still very difficult to study 4-dimensional gauge theories, we would like to start from $1+1$ dimensions. In $1+1$ dimensions, gauge fields have no dynamical degrees of freedom. If matter fields belong to the fundamental representation of a gauge group, they become tractable in the $1/N$ approximation, and provide an illuminating model for confining theories [3]. If there are matter fields in adjoint representations, the $1/N$ approximation is not sufficient to solve the $SU(N)$ gauge theories. Still a number of recent studies have been performed both numerically and analytically for Yang-Mills theories with adjoint matter fields, and the studies have yielded several interesting results [6]-[14]. The Born-Oppenheimer approximation in the weak coupling region has been used to study the vacuum structure of gauge theories with adjoint fermions [12]. Since the gauge coupling in $1 + 1$ dimensions has the dimension of mass, the weak coupling is characterized by

$$gL \ll 1, \tag{1.1}$$

where L is the interval of the compactified spatial dimension. The fermion bilinear was found to possess a nonvanishing vacuum expectation value which exhibits instanton-like dependence on gauge coupling. The Yang-Mills theories with adjoint fermions were also studied at finite temperature and were shown to be dominated by instanton effects at high temperatures [10]. The Born-Oppenheimer approximation has been used to study SUSY gauge theories in four dimensions [15][16].

Since the gauge fields have no dynamical degree of freedom, SUSY Yang-Mills theories in $1 + 1$ dimensions (SYM$_2$) contain both spinor field (gaugino) and scalar field in the adjoint representation. A manifestly supersymmetric (infrared) regularization scheme has been obtained recently using the discretized light-cone approach [5]. In this study, numerical results suggested an exponentially rising density of states. Our understanding of these theories is, however, not yet sufficient. In particular, vacuum structures such as the vacuum condensate need to be clarified. We need alternative systematic approaches to study them thoroughly, since the light-cone approach which is best suited to deal with excited states is notoriously laborious when applied to the unraveling of vacuum structures. The concept of zero modes is crucial in understanding vacuum structures [17], [18]. As for the possibility of the SUSY breaking, the Witten index of the SUSY Yang-Mills theories has been calculated recently, and was found to be nonvanishing

[4]. Although this result implies no possibility for spontaneous SUSY breaking, we feel it still worthwhile to study the vacuum of the SUSY Yang-Mills theories in $1+1$ dimensions by a more detailed dynamical calculation, since the calculation of the Witten index involved a certain regularization of bosonic zero modes which may not be easily justified.

The purpose of this paper is to study vacuum structures of supersymmetric Yang-Mills theories in 1+1 dimensions. We use the Born-Oppenheimer approximation in the weak coupling region, as used for non-SUSY Yang-Mills theories with adjoint fermions [12]. To formulate the weak coupling limit, we need to compactify the spatial direction. Since gauge fields naturally follow periodic boundary conditions, we need to require the same periodic boundary conditions for scalar and spinor fields in order not to break SUSY by hand. We have found that the ground state has a vanishing vacuum energy, suggesting that SUSY is not broken spontaneously. This result is consistent with the result on the Witten index [4]. We also examine all four possibilities of periodic and anti-periodic boundary conditions for fermions and bosons. The case involving antiperiodic fermions and periodic bosons should be related to the case of finite temperature. We find in this case that the vacuum energy does not vanish, and the gaugino bilinear exhibits nonvanishing vacuum condensation. The vacuum condensate turns out to have nontrivial dependence on the dimensionless constant gL, which resembles the instanton contributions. It would be interesting to conduct a study to see if our results can be explained by instanton contributions.

Sec. II briefly summarizes the Hamiltonian approach to SUSY Yang-Mills theories in $1+1$ dimensions. The canonical quantization is carried out, and dynamical degrees of freedom are identified. In Sec. III, vacuum structures of SYM_2 are discussed using an adiabatic approximation. Sec. IV discusses cases involving antiperiodic boundary conditions for spinors or scalars, and Sec. V contains a summary.

2. SUSY Yang-Mills Theories in $1+1$ Dimensions

Since gauge fields have no dynamical degrees of freedom in two dimensions, the SUSY gauge multiplet in 1+1 dimensions consists of gauge fields A^μ, the Majorana spinor Ψ, and the real scalar ϕ [20]. SUSY $SU(N)$ Yang-Mills action is given by Ψ and ϕ fields in the adjoint representation as $\phi \equiv \phi^a t^a$ and $\Psi \equiv \Psi^a t^a$, where the t^a are generators of $SU(N)$ with the normalization $\mathrm{tr}\left(t^a t^b\right) = \frac{1}{2}\delta^{ab}$ [20]

$$\mathcal{L} = \mathrm{tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi D^\mu\phi + i\bar{\Psi}\gamma^\mu D_\mu\Psi - ig\phi\bar{\Psi}\gamma_5\Psi\right\}. \tag{2.1}$$

The gauge coupling is denoted as g, $D_\mu = \partial_\mu + ig[A_\mu, \ ]$ is the usual covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. The action is invariant under the following supersymmetry

3

transformations [20] (also see [5]).

$$\delta A_\mu = i\bar\epsilon\gamma_5\gamma_\mu\Psi, \quad \delta\phi = -\bar\epsilon\Psi, \quad \delta\Psi = -\frac{1}{2}\epsilon\epsilon^{\mu\nu}F_{\mu\nu} + i\gamma^\mu\epsilon D_\mu\phi. \quad (\epsilon^{01} = -\epsilon_{01} = 1) \qquad (2.2)$$

Taking the following representation of γ matrices

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma^5 \equiv \gamma^0\gamma^1 = \sigma_3, \quad C = -\sigma_2, \qquad (2.3)$$

the Majorana spinor $\psi = C\bar\psi^T$ is real.

In this paper we compactify the spatial direction to a circle with a finite radius $L/2\pi$. The gauge fields naturally follow periodic boundary conditions

$$A^\mu(x = 0) = A^\mu(x = L). \qquad (2.4)$$

We shall specify boundary conditions for Ψ and ϕ later.

Gauge theories have a large number of redundant gauge degrees of freedom which should be eliminated by a gauge-fixing condition. In this paper we quantize the system in the Weyl gauge,

$$A^0 = 0. \qquad (2.5)$$

We can impose Gauss' law as a subsidiary condition for the physical state $|\Phi\rangle$

$$[D_1 E^a(x) - g\rho^a(x)]|\Phi\rangle = 0, \quad \rho^a = f^{abc}\phi^b\pi^c + \frac{i}{2}f^{abc}\Psi_\alpha^c\Psi_\alpha^b. \qquad (2.6)$$

where π^a and $-E^a \equiv F^{a01}$ are the conjugate variables of ϕ^a and A^{a1} respectively, and ρ^a is the color charge density, and f^{abc} are the structure constants of the Lie algebra of $SU(N)$: $[t^a, t^b] = if^{abc}t^c$. Note that the Gauss law determines E, except for its constant modes e. One can eliminate A^1 by using an appropriate gauge transformation, except for the $N-1$ spatially constant modes a^p which are given by

$$\mathcal{P}\exp\left(ig\int_0^L dx A^1(x)\right) = Ve^{igaL}V^\dagger, \quad (a = a^pt^p), \qquad (2.7)$$

where V is a unitary matrix. Hereafter we shall use the convention that $a, b, \cdots = 1, 2\cdots, N^2-1$ represent the indices of the generators of $SU(N)$, and $p, q, \cdots = 1, 2\cdots, N-1$ represent those of Cartan subalgebra. The commutation relation between a^p and e^q is given as [11]

$$[e^p, a^q] = i\delta^{pq} \qquad p, q = 1, \ldots, N-1. \qquad (2.8)$$

In the physical state space, we can eliminate redundant gauge degrees of freedom by solving the Gauss law constraint (2.6), and find the Hamiltonian

$$H = \int_0^L dx\mathcal{H}(x) = K_a + H_c + H_b + H_f + H_{\text{int}}, \qquad (2.9)$$

4

$$K_a = \frac{1}{2L} \sum_p e^{p\dagger} e^p, \tag{2.10}$$

$$H_c = \frac{g^2}{L} \sum_{n=-\infty}^{\infty} \sum_{ij} \int_0^L dy \int_0^L dz (1 - \delta_{ij}\delta_{n0}) \frac{(\rho(y))_{ij} (\rho(z))_{ji}}{\left(\frac{2\pi n}{L} + g(a_i - a_j)\right)^2} e^{2\pi in(y-z)/L}, \tag{2.11}$$

$$H_b = \int_0^L dx \left\{ \frac{1}{2}\pi^a\pi^a + \frac{1}{2}(D_1\phi)^a (D_1\phi)^a \right\}, \tag{2.12}$$

$$H_f = \int_0^L dx \left(-\frac{i}{2}\right) \Psi^a \sigma_3 (D_1\Psi)^a, \qquad H_{\text{int}} = \int_0^L dx \, \text{tr} \left\{ ig\phi\bar{\Psi}\gamma_5\Psi \right\} \tag{2.13}$$

where $a_i = a^p t_{ii}^p$ with no summation over i implied, and $\sum_i a_i = 0$. Here the covariant derivative D_1 contains only the zero mode of $A^1 : D_1 = \partial_1 - ig[a,]$. One should note that gauge fields A^μ, except the zero modes a^p, are completely eliminated.

In order to investigate the vacuum structures of our model, we solve Schrödinger's equation with respect to the Hamiltonian (2.9)

$$H|\Phi\rangle = E|\Phi\rangle, \tag{2.14}$$

where $|\Phi\rangle$ denote state vectors in the physical space. Because of hermiticity of the variables a, the kinetic energy K_a is given in terms of the Jacobian $J[a]$ of the transformation (2.7) [11]

$$K_a = \frac{1}{2L}e^{p\dagger}e^p = -\frac{1}{2L}\frac{1}{J[a]}\frac{\partial}{\partial a^p}J[a]\frac{\partial}{\partial a^p}, \tag{2.15}$$

$$J[a] = \prod_{i>j} \sin^2\left(\frac{1}{2}gL(a_i - a_j)\right). \tag{2.16}$$

In analogy with the radial wavefunctions, it is useful to define a modified wave function

$$\tilde{\Phi}[a] \equiv \sqrt{J[a]}\Phi[a]. \tag{2.17}$$

The kinetic energy operator for $\tilde{\Phi}$ is (with the notation $\partial_p = \partial/\partial a^p$),

$$K'_a \equiv \sqrt{J}K_a\frac{1}{\sqrt{J}} = -\frac{1}{2L}\partial_p\partial_p + V^{[N]}, \tag{2.18}$$

$$V^{[N]} \equiv \frac{1}{2L}\frac{1}{\sqrt{J}}\left(\partial_p\partial_p\sqrt{J}\right) = -\frac{(gL)^2}{48L}N(N^2 - 1). \tag{2.19}$$

Thus we obtain a boundary condition for the modified wavefunction,

$$\tilde{\Phi}[a] = 0, \quad \text{if} \quad J[a] = 0 . \tag{2.20}$$

Let us now quantize the fields Ψ and ϕ. The gauge field zero modes a^p couple only to off-diagonal elements, which are parameterized as : $\varphi_{ij} = \sqrt{2}\Psi_{ij}$, $\varphi_{ij}^\dagger = \sqrt{2}\Psi_{ji}$, $\xi_{ij} = \sqrt{2}\phi_{ij}$,

$\xi_{ij}^\dagger = \sqrt{2}\phi_{ji}$, $\eta_{ij} = \sqrt{2}\pi_{ij}$, and $\eta_{ij}^\dagger = \sqrt{2}\pi_{ji}$ ($i < j$). With these conventions the Hamiltonian takes the form

$$H_{\mathrm{f}} = H_{\mathrm{f,diag}} + H_{\mathrm{f,off}}, \qquad H_{\mathrm{b}} = H_{\mathrm{b,diag}} + H_{\mathrm{b,off}}, \tag{2.21}$$

$$H_{\mathrm{f,diag}} = \frac{1}{2i} \sum_p \int_0^L dx \, \Psi^p \sigma_3 \partial_1 \Psi^p, \tag{2.22}$$

$$H_{\mathrm{f,off}} = \sum_{i<j} \int_0^L dx \, \varphi_{ij}^\dagger \sigma_3 \left(\frac{1}{i}\partial_1 - g(a_i - a_j) \right) \varphi_{ij}, \tag{2.23}$$

$$H_{\mathrm{b,diag}} = \sum_p \int_0^L dx \left(\frac{1}{2}\pi^p \pi^p + \frac{1}{2}\left(\partial_1 \phi^p \right)\left(\partial_1 \phi^p \right) \right), \tag{2.24}$$

$$H_{\mathrm{b,off}} = \sum_{i<j} \int_0^L dx \Big\{ \eta_{ij}^\dagger \eta_{ij} + \left(\partial_1 \xi_{ij}^\dagger - ig(a_j - a_i)\xi_{ij}^\dagger \right)\left(\partial_1 \xi_{ij} - ig(a_i - a_j)\xi_{ij} \right) \Big\}. \tag{2.25}$$

Let us now discuss the range of the variables a^p [12]. Eq.(2.7) shows that the gLa are angular variables which are defined only in modulo 2π. If the parameterization of a is one-to-one and permutations of the eigenvalues are contained in a single domain, the domain is called *the elementary cell*. For example, in the SU(2) case, two eigenvalues of the matrix a are $a_1 = a^3/2$ and $a_2 = -a^3/2$. Then, the elementary cell is the interval $-\pi \leq \frac{gLa^3}{2} \leq \pi$, with the end points identified. If a^3 is negative in the elementary cell, the Weyl reflection $a^3 \to -a^3$ maps the interval $-\frac{2\pi}{gL} < a^3 < 0$ onto the interval $0 < a^3 < \frac{2\pi}{gL}$ (simultaneously, $\varphi^{12} \leftrightarrow \varphi^{21}$). In the $SU(N)$ case, similarly, the elementary cell is divided into $N!$ domains by the Weyl group since the Weyl group of $SU(N)$ is the permutation group P_N. These $N!$ domains are called *fundamental domains*. Boundaries of the fundamental domains consist of the hypersurfaces where two of the eigenvalues match. If two of the eigenvalues have the same value, the Jacobian $J[a]$ vanishes. In the case of $SU(2)$, we take the following interval as the fundamental region

$$0 \leq a^3 \leq \frac{2\pi}{gL}. \tag{2.26}$$

The Jacobian $J[a] = \sin^2\left(\frac{1}{2}gLa^3 \right)$ vanishes at $a^3 = 0$, $\frac{2\pi}{gL}$. Note that the modified wavefunction $\tilde{\Phi}[a]$ vanishes at these points.

3. Vacuum Structures of SUSY $SU(2)$ Yang-Mills Theories

In this section, we determine the wave function of the vacuum state in the fundamental domain by using the Born-Oppenheimer approximation [12]. If $gL \ll 1$, the energy scale of the system of a^p is given by $(gL)^2/L$, while that of non-zero modes of Ψ and ϕ is in general of

order $1/L$. Therefore we can integrate the non-zero modes of Ψ and ϕ to obtain the effective potential for a^p. We will retain the zero modes of Ψ and ϕ, since their spectrum is continuous. By solving the Schrödinger equation with respect to the resulting effective potential, we obtain the wavefunction $\tilde{\Phi}[a]$, which describes the vacuum structures of our model. In these procedures we must pay attention to the boundary conditions for $\tilde{\Phi}[a]$ resulting from the Jacobian (2.20).

To calculate the effective potential as a function of the gauge zero modes a^p, we have to find the ground state of fermion Ψ and boson ϕ for a fixed value of a^p. Here, we must take care with regards to the boundary conditions for $\Psi(x)$ and $\phi(x)$. Since spinors and scalars are superpartners of gauge fields which obey the periodic boundary condition, the spinors $\Psi(x)$ and scalars $\phi(x)$ should be periodic in order for the boundary conditions to maintain supersymmetry

$$\Psi(x = 0) = \Psi(x = L), \qquad \phi(x = 0) = \phi(x = L). \tag{3.1}$$

Hereafter we refer to this boundary condition as *the (P,P) case*. In this section we investigate the vacuum structures for the gauge group $SU(2)$. We will discuss other boundary conditions later.

3.1. Born-Oppenheimer Approximation

For $gL \ll 1$, the Coulomb energy (2.11) and the Yukawa interaction (2.13) can be neglected. In this limit, the relevant parts of the Hamiltonian are, for $SU(2)$,

$$\tilde{H} = K'_a + H_{\text{b,diag}} + H_{\text{b,off}} + H_{\text{f,diag}} + H_{\text{f,off}}. \tag{3.2}$$

$$K'_a = -\frac{1}{2L}\frac{\partial^2}{\partial a^2} + V^{[N=2]}, \tag{3.3}$$

$$H_{\text{b,diag}} = \frac{1}{2}\int_0^L dx \left\{ \pi^3\pi^3 + (\partial_1\phi^3)(\partial_1\phi^3) \right\} \tag{3.4}$$

$$H_{\text{b,off}} = \int_0^L dx \left\{ \eta^\dagger\eta + (\partial_1\xi^\dagger + iga\xi^\dagger)(\partial_1\xi - iga\xi) \right\}, \tag{3.5}$$

$$H_{\text{f,diag}} = \frac{1}{2i}\int_0^L dx \Psi^3\sigma_3\partial_1\Psi^3, \qquad H_{\text{f,off}} = \int_0^L dx \varphi^\dagger\sigma_3\left(\frac{1}{i}\partial_1 - ga\right)\varphi, \tag{3.6}$$

$$\varphi \equiv \varphi_{12}, \quad \xi \equiv \xi_{12}, \quad \eta \equiv \eta_{12}, \quad a \equiv a^3 = a_1 - a_2. \tag{3.7}$$

A remnant of large gauge transformations becomes a discrete symmetry S [12]

$$S: \qquad a \to -a + \frac{2\pi}{gL},$$
$$\varphi \to e^{2i\pi x/L}\varphi^\dagger, \quad \xi \to e^{2i\pi x/L}\xi^\dagger, \quad \eta \to e^{2i\pi x/L}\eta^\dagger,$$
$$\Psi^3 \to -\Psi^3, \quad \phi^3 \to -\phi^3, \quad \pi^3 \to -\pi^3. \tag{3.8}$$

This operator can be chosen to satisfy $S^2 = 1$ and $[S, H] = 0$. SYM$_2$ has a topologically nontrivial structure $\pi_1[SU(N)/Z_N] = Z_N$. The symmetry S corresponds to a nontrivial element of this $Z_{N=2}$ group for $SU(2)$.

In order to perform the Born-Oppenheimer approximation, we first expand the spinor fields φ and Ψ^3, and impose a canonical anticommutation relation

$$\varphi(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} e^{i2\pi kx/L}, \qquad \{a_k, a_{k'}^\dagger\} = \{b_k, b_{k'}^\dagger\} = \delta_{k,k'},$$

$$\Psi^3(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} \begin{pmatrix} c_k \\ d_k \end{pmatrix} e^{i2\pi kx/L}, \qquad c_{-k} = c_k^\dagger, \quad d_{-k} = d_k^\dagger,$$

$$\{c_k, c_{k'}^\dagger\} = \{d_k, d_{k'}^\dagger\} = \delta_{k,k'}, \quad k, k' \geq 0 \tag{3.9}$$

The Hamiltonian $H_{\text{f,off}}$ in (3.6) takes the form

$$H_{\text{f,off}} = \sum_{k=-\infty}^{\infty} \left(a_k^\dagger a_k - b_k^\dagger b_k\right) \left(\frac{2\pi k}{L} - ga\right). \tag{3.10}$$

In the Born-Oppenheimer approximation, the vacuum state for the off-diagonal part of the fermion is obtained by filling the Dirac sea for the fermion φ. We assume the a_k modes to be filled for $k < M$. The Gauss law constraint (2.6) dictates that the b_k modes should be filled for $k \geq M$ [12]. Denoting the vacuum state for the fermion as $|0_\varphi; M\rangle$, the vacuum energy can be written as

$$H_{\text{f,off}}|0_\varphi; M\rangle = \left[\sum_{k=-\infty}^{M-1} \left(\frac{2\pi k}{L} - ga\right) - \sum_{k=M}^{\infty} \left(\frac{2\pi k}{L} - ga\right)\right] |0_\varphi; M\rangle$$

$$\equiv V_{\text{f,off}}(a; M)|0_\varphi; M\rangle. \tag{3.11}$$

Notice that S acts on the state $|0_\varphi; M\rangle$ according to

$$S|0_\varphi; M\rangle = e^{i\alpha_M}|0_\varphi; 2 - M\rangle. \tag{3.12}$$

In addition, the phase factor $e^{i\alpha_M}$ is constrained by $S^2 = 1$, or in other words, $e^{i\alpha_M} = e^{-i\alpha_{-M+2}}$.

For diagonal part of the fermion, we obtain the Hamiltonian from (3.6)

$$H_{\text{f,diag}} = \sum_{k\geq 1} \frac{2\pi k}{L} \left(c_k^\dagger c_k + d_k d_k^\dagger - 1\right). \tag{3.13}$$

On the vacuum $|0_\Psi\rangle$ defined by $c_k|0_\Psi\rangle = d_k^\dagger|0_\Psi\rangle = 0$, $k \geq 1$, we find

$$H_{\text{f,diag}}|0_\Psi\rangle = -\sum_{k\geq 1} \frac{2\pi k}{L}|0_\Psi\rangle \equiv V_{\text{f,diag}}|0_\Psi\rangle. \tag{3.14}$$

Next we expand the scalar fields ξ, η, ϕ^3, and π^3, and impose canonical commutation relations

$$\xi(x) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2LE_k}} \left(e_k + f_k^\dagger\right) e^{i2\pi kx/L}, \quad E_k = \left|\frac{2\pi k}{L} - ga\right|, \tag{3.15}$$

$$\eta(x) = \sum_{k=-\infty}^{\infty} i\sqrt{\frac{E_k}{2L}} \left(-e_k + f_k^\dagger\right) e^{i2\pi kx/L}, \tag{3.16}$$

$$\phi^3(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{\sqrt{2LF_k}} \left(g_k + g_{-k}^\dagger\right) e^{i2\pi kx/L} + \phi_{\text{zero}}, \quad F_k = \left|\frac{2\pi k}{L}\right|, \tag{3.17}$$

$$\pi^3(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} i\sqrt{\frac{F_k}{2L}} \left(-g_k + g_{-k}^\dagger\right) e^{i2\pi kx/L} + \frac{1}{L}\pi_{\phi_{\text{zero}}}. \tag{3.18}$$

$$\left[e_k, e_{k'}^\dagger\right] = \left[f_k, f_{k'}^\dagger\right] = \left[g_k, g_{k'}^\dagger\right] = \delta_{k,k'}, \quad [\phi_{\text{zero}}, \pi_{\phi_{\text{zero}}}] = i. \tag{3.19}$$

The Hamiltonian $H_{\text{b,off}}$ in (3.5) is given by

$$H_{\text{b,off}} = \sum_{k=-\infty}^{\infty} E_k \left(e_k^\dagger e_k + f_k f_k^\dagger\right) \tag{3.20}$$

$$= \sum_{k=-\infty}^{\infty} E_k \left(e_k^\dagger e_k + f_k^\dagger f_k\right) - \sum_{k=-\infty}^{N-1} \left(\frac{2\pi k}{L} - ga\right) + \sum_{k=N}^{\infty} \left(\frac{2\pi k}{L} - ga\right), \tag{3.21}$$

where N is an integer satisfying

$$\frac{2\pi N}{L} - ga \geq 0, \qquad \frac{2\pi(N-1)}{L} - ga < 0. \tag{3.22}$$

On the vacuum state $|0_\xi\rangle$ defined by $e_k|0_\xi\rangle = f_k|0_\xi\rangle = 0$, for all k, we find the vacuum energy

$$\begin{aligned} H_{\text{b,off}}|0_\xi\rangle &= \left[-\sum_{k=-\infty}^{N-1} \left(\frac{2\pi k}{L} - ga\right) + \sum_{k=N}^{\infty} \left(\frac{2\pi k}{L} - ga\right)\right] |0_\xi\rangle \\ &\equiv V_{\text{b,off}}(a)|0_\xi\rangle. \end{aligned} \tag{3.23}$$

We find that the zero mode Hamiltonian H_0 is separated as

$$H_{\text{b,diag}} = \sum_{k \geq 1} \frac{2\pi k}{L} \left(g_k^\dagger g_k + g_{-k}^\dagger g_{-k} + 1\right) + H_0, \tag{3.24}$$

$$H_0 = \frac{1}{2L}\pi_{\phi_{\text{zero}}}\pi_{\phi_{\text{zero}}}. \tag{3.25}$$

9

On the vacuum for the nonzero modes of ϕ^3 satisfying $g_k|0_\phi\rangle = g_{-k}|0_\phi\rangle = 0, k \geq 1$, we find the vacuum energy

$$V_{\text{b,diag}} = \sum_{k \geq 1} \frac{2\pi k}{L}. \tag{3.26}$$

3.2. Vacuum Structure

The vacuum energies obtained in the previous section are divergent. By regularizing them with the heat kernel, we obtain the following finite effective potential as a function of a

$$
\begin{aligned}
U_{M,N}(a) &= V_{\text{f,off}}(a; M) + V_{\text{b,off}}(a) + V_{\text{f,diag}} + V_{\text{b,diag}} + V^{[N=2]} \\
&= \frac{2\pi}{L} \left(M - \frac{gLa}{2\pi} - \frac{1}{2} \right)^2 - \frac{2\pi}{L} \left(N - \frac{gLa}{2\pi} - \frac{1}{2} \right)^2 + V^{[N=2]}.
\end{aligned} \tag{3.27}
$$

In the fundamental region $0 < \frac{gLa}{2} < \pi$, $N = 1$ from (3.22). By requiring that the vacuum energy $U_{M,N}(a)$ be minimal, we can fix M to obtain $M = 1$. We then find that the total vacuum energy in the fundamental domain is independent of a

$$U_{M,N}(a) = V^{[N=2]}. \tag{3.28}$$

Consequently we obtain the Hamiltonian which describes the vacuum structures for the periodic boundary condition

$$\tilde{H} = K_a' + H_0 = -\frac{1}{2L} \frac{\partial}{\partial a} \frac{\partial}{\partial a} + V^{[N=2]} + \frac{1}{2L} \pi_{\phi_{\text{zero}}} \pi_{\phi_{\text{zero}}}. \tag{3.29}$$

We also have the zero modes of the fermion, which form a Clifford algebra

$$\Psi^3_{\text{zero}} = \frac{1}{\sqrt{L}} \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}, \qquad c_0 = c_0^\dagger, \quad d_0 = d_0^\dagger, \tag{3.30}$$

$$\left\{ \lambda, \lambda^\dagger \right\} = 1, \quad \{\lambda, \lambda\} = \left\{ \lambda^\dagger, \lambda^\dagger \right\} = 0, \quad \lambda \equiv \frac{1}{\sqrt{2}} (c_0 + id_0). \tag{3.31}$$

Let us now solve the Schrödinger equation

$$\tilde{H}\tilde{\Phi}(a) = e\tilde{\Phi}(a). \tag{3.32}$$

Because of the boundary condition (2.20) we get the wavefunction $\tilde{\Phi}(a)$ and the energy eigenvalue e of the ground state as

$$\tilde{\Phi}(a) = \sqrt{\frac{gL}{\pi}} \sin\left(\frac{gLa}{2} \right), \qquad e = 0. \tag{3.33}$$

10

It is interesting to note that the vacuum energy associated with the nontrivial zero mode wave-function (3.33) cancels precisely the contribution $V^{[N=2]}$ from the Jacobian in (2.19). Therefore we have shown explicitly that the SUSY is not broken spontaneously. Also note that our result is consistent with the previous calculation of the nonvanishing Witten index [4]. The calculation, however, ignores the Jacobian (2.16), which is an important ingredient in our present attempt to define the gauge field zero modes properly [12]. Therefore the above explicit demonstration of the vanishing vacuum energy using the Born-Oppenheimer approximation can be regarded as another independent proof of the unbroken SUSY in SUSY Yang-Mills theories in $1+1$ dimensions.

We define the vacuum state of the zero modes of the fermion c_0, d_0. Note that the zero modes belong to the two-dimensional representation of the Clifford algebra (3.31). We define $|\Omega\rangle$ to be the Clifford vacuum annihilated by λ and $|\tilde{\Omega}\rangle = \lambda^\dagger|\Omega\rangle$. Since the field ϕ^3 can take unbounded values, the zero mode spectrum is continuous. This fact makes the Witten index ill-defined. The previous attempt to compute the Witten index employed a regularization by putting a cut-off on the ϕ_{zero} space. In that case, the Witten index can be defined and obtains $\text{tr}(-1)^F = 1$ [4]. In spite of this complication, we can choose the wave function to be constant in the ϕ^3 zero mode as the vacuum: $H_0|\omega\rangle = 0$.

Let us now examine the transformation property under the discrete gauge transformation S. The non-zero mode vacuum $|0_\varphi; M = 1\rangle$ turns out to be an eigenstate of S

$$S|0_\varphi; M = 1\rangle = \pm|0_\varphi; M = 1\rangle \tag{3.34}$$

because of eq.(3.12) and $S^2 = 1$. Similarly $|0_\Psi\rangle$, $|0_\xi\rangle$, $|0_\phi\rangle$ and $|\omega\rangle$ are eigenstate of S with eigenvalues ± 1. For the fermion zero mode, $S|\Omega\rangle = \pm|\Omega\rangle$ and $S|\tilde{\Omega}\rangle = \mp|\tilde{\Omega}\rangle$. Since we should construct the full vacuum state as an eigenstate with eigenvalue ± 1 for S

$$|\mathbf{0}_\Omega\rangle \equiv |\tilde{\Phi}(a)\rangle|0_\varphi; M = 1\rangle|0_\Psi\rangle|0_\xi\rangle|0_\phi\rangle|\omega\rangle|\Omega\rangle,$$
$$|\mathbf{0}_{\tilde{\Omega}}\rangle \equiv |\tilde{\Phi}(a)\rangle|0_\varphi; M = 1\rangle|0_\Psi\rangle|0_\xi\rangle|0_\phi\rangle|\omega\rangle|\tilde{\Omega}\rangle. \tag{3.35}$$

We find the vacuum condensate $\left|\langle\mathbf{0}|\bar{\Psi}^a\Psi^a|\mathbf{0}\rangle\right| = \frac{1}{L}$ for both $|\mathbf{0}\rangle = |\mathbf{0}_\Omega\rangle$ and $|\mathbf{0}_{\tilde{\Omega}}\rangle$. One can see that this condensate is due to the finite spacial extent L.

4. Cases with Other Boundary Conditions

In this section, we study other boundary conditions for the fermions Ψ and the bosons ϕ. There are four cases, depending on the choice of periodic or antiperiodic boundary conditions

	Fermion b. c.	Boson b. c.
(P,P) case (SUSY)	periodic	periodic
(A,P) case	antiperiodic	periodic
(A,A) case	antiperiodic	antiperiodic
(P,A) case	periodic	antiperiodic

11

We shall study in turn the (A, P), (A, A), and (P, A) cases, and will find that the vacuum energy does not vanish. This indicates that SUSY is broken by boundary conditions in these three cases.

4.1. The (A, P) Case; AntiPeriodic Fermion and Periodic Boson

We first discuss the (A, P) case, where the following boundary conditions are imposed on the fermions Ψ and bosons ϕ

$$\Psi^a(x = 0) = -\Psi^a(x = L), \quad \phi^a(x = 0) = \phi^a(x = L). \tag{4.1}$$

One of the motivations for considering this case is that one can naturally regard L as the inverse temperature in the finite-temperature situation.

Let us first consider the fermionic parts of the Hamiltonian H_f. We expand the spinor fields φ and Ψ^3 into modes, and obtain the Hamiltonian $H_\mathrm{f,off}$ in (3.6)

$$\varphi(x) = \frac{1}{\sqrt{L}} \sum_{k \in \mathbf{Z}} \begin{pmatrix} a_k \\ b_k \end{pmatrix} e^{i2\pi(k+1/2)x/L}, \tag{4.2}$$

$$H_\mathrm{f,off} = \sum_{k \in \mathbf{Z}} \left(a_k^\dagger a_k - b_k^\dagger b_k \right) \left(\frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right). \tag{4.3}$$

Similarly to the (P, P) case, the vacuum state for the off-diagonal part of the fermion is obtained by filling the fermion negative energy states. We assume the a_k modes to be filled for $k < M$ and the b_k modes for $k \geq M$. Then the vacuum energy of $H_\mathrm{f,off}$ is given by

$$
\begin{aligned}
H_\mathrm{f,off} |0_\varphi; M\rangle &= \left[\sum_{k=-\infty}^{M-1} \left(\frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right) - \sum_{k=M}^{\infty} \left(\frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right) \right] |0_\varphi; M\rangle \\
&\equiv V_\mathrm{f,off}(a; M) |0_\varphi; M\rangle.
\end{aligned} \tag{4.4}
$$

The symmetry operator S is defined as follows

$$S|0_\varphi; M\rangle = e^{i\alpha_M} |0_\varphi; 1 - M\rangle, \qquad e^{i\alpha_M} = e^{-i\alpha_{-M+1}}. \tag{4.5}$$

As for $H_\mathrm{f,diag}$ in (3.6), we obtain

$$\Psi^3(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{\infty} \begin{pmatrix} c_k \\ d_k \end{pmatrix} e^{i2\pi(k+1/2)x/L}, \quad c_{-k-1} = c_k^\dagger, \quad d_{-k-1} = d_k^\dagger. \tag{4.6}$$

$$H_\mathrm{f,diag} = \sum_{k \geq 0} \frac{2\pi \left(k + \frac{1}{2} \right)}{L} \left(c_k^\dagger c_k + d_k d_k^\dagger - 1 \right). \tag{4.7}$$

12

Therefore the vacuum state satisfying $c_k|0_\Psi\rangle = d_k^\dagger|0_\Psi\rangle = 0$, $(k \geq 0)$ has energy

$$H_{\text{f,diag}}|0_\Psi\rangle = -\sum_{k\geq 0}\frac{2\pi\left(k+\frac{1}{2}\right)}{L}|0_\Psi\rangle \equiv V_{\text{f,diag}}|0_\Psi\rangle. \tag{4.8}$$

As for the bosonic part of the Hamiltonian H_{b} in the (A, P) case, we may use the result of the (P, P) case because the boundary conditions for the scalar fields are the same.

The heat kernel regularized potential (3.27) for the gauge field zero mode a becomes in this case

$$U_{M,N}(a) = \frac{2\pi}{L}\left(M - \frac{gLa}{2\pi}\right)^2 - \frac{2\pi}{L}\left(N - \frac{1}{2} - \frac{gLa}{2\pi}\right)^2 - \frac{\pi}{4L} + V^{[N=2]}. \tag{4.9}$$

Here, as in the (P, P) case, $N = 1$. We can think of this as a potential in quantum mechanics for the zero mode a. Note that M should be chosen by requiring that the ground state energy of the (A, P) case be minimal. Eq.(4.9) gives two solutions; $M = 0, 1$. Let us first consider zero mode quantum mechanics in the $M = 1$ case (see Fig. 1)

$$\tilde{H}\tilde{\Phi}_{\text{II}} = e\tilde{\Phi}_{\text{II}}, \qquad \tilde{H} = -\frac{1}{2L}\frac{\partial^2}{\partial a^2} + V_{M=1}(a) + H_0. \tag{4.10}$$

$$V_{M=1}(a) = \begin{cases} \infty, & \text{when } a = 0 \\ U_{1,1}(a) = -ga + \frac{5\pi}{4L} + V^{[N=2]}, & \text{when } 0 < a < 2\pi/gL \\ \infty, & \text{when } a = 2\pi/gL \end{cases} \tag{4.11}$$

The Hamiltonian for the zero mode of the scalar field is H_0 in (3.25).

We find eigenfunctions with a normalization factor A_+ and A_-

$$\tilde{\Phi}_{\text{II}}(a) = \begin{cases} \xi_2^{\frac{1}{3}}(a)\left\{-A_+I_{\frac{1}{3}}\left(\xi_2(a)\right) + A_-I_{-\frac{1}{3}}\left(\xi_2(a)\right)\right\}, & \text{when } 0 \leq a \leq s \\ \xi_1^{\frac{1}{3}}(a)\left\{A_+J_{\frac{1}{3}}\left(\xi_1(a)\right) + A_-J_{-\frac{1}{3}}\left(\xi_1(a)\right)\right\}, & \text{when } s < a \leq 2\pi/gL \end{cases} \tag{4.12}$$

$$\xi_1(a) \equiv \frac{2}{3}\sqrt{2gL}\,(a-s)^{\frac{3}{2}}, \quad \xi_2(a) \equiv \frac{2}{3}\sqrt{2gL}\,(-a+s)^{\frac{3}{2}}, \tag{4.13}$$

$$s = -\frac{1}{g}\left(e - \frac{5\pi}{4L} - V^{[N=2]}\right), \tag{4.14}$$

where J_ν are Bessel functions and I_ν are modified Bessel functions. Imposing the boundary conditions (2.20) on the wavefunction, we obtain the eigenvalues of $\tilde{H}$ as

$$e_n = -\frac{3\pi}{4L} + \frac{1}{2L}\left(3b_ngL\right)^{\frac{2}{3}} - \frac{1}{8L}(gL)^2, \qquad n \in \mathbf{N}, \tag{4.15}$$

where the b_n are defined as

$$I_{-\frac{1}{3}}(\xi_1(0))J_{\frac{1}{3}}(b_n) + I_{\frac{1}{3}}(\xi_1(0))J_{-\frac{1}{3}}(b_n) = 0, \qquad b_n > 0. \tag{4.16}$$

13

Hence the ground state energy does not vanish. In the $M = 0$ case, on the other hand, the potential is given by $V_{M=0}(a) = U_{\{M=0,\ N=1\}}(a) = U_{\{M=1,\ N=1\}}(2\pi/gL - a)$. It then follows that the modified wavefunction $\tilde{\Phi}_{\mathrm{I}}(a)$ in the $M = 0$ case is

$$\tilde{\Phi}_{\mathrm{I}}(a) = \tilde{\Phi}_{\mathrm{II}}(2\pi/gL - a). \tag{4.17}$$

As discussed in [12], the full vacuum state is determined by requiring that it should be an eigenstate of the symmetry S, which acts as

$$S|0_\varphi; M\rangle = e^{i\alpha_M}|0_\varphi; 1 - M\rangle, \qquad S\tilde{\Phi}_{\mathrm{I}}(a) = \tilde{\Phi}_{\mathrm{II}}(a). \tag{4.18}$$

It follows from this that the full vacuum state $|\mathbf{0}_\pm\rangle$ is given by superposing the two states $M = 0, 1$

$$
\begin{aligned}
|\mathbf{0}_\pm\rangle &\equiv \frac{1}{\sqrt{2}}\left(|\tilde{\Phi}_{\mathrm{I}}(a)\rangle|0_\varphi; M = 0\rangle \pm e^{i\alpha_0}|\tilde{\Phi}_{\mathrm{II}}(a)\rangle|0_\varphi; M = 1\rangle\right) \\
&\otimes |0_\Psi\rangle|0_\xi\rangle|0_\phi\rangle|\omega\rangle.
\end{aligned} \tag{4.19}
$$

$|\mathbf{0}_\pm\rangle$ have eigenvalues ± 1 of S. Here we assume that $|0_\Psi\rangle|0_\xi\rangle|0_\phi\rangle|\omega\rangle$ is invariant under S.

Let us consider the gaugino bilinear condensate in this vacuum $|\mathbf{0}_\pm\rangle$.

$$\langle \mathbf{0}_\pm|\bar{\Psi}^a\Psi^a|\mathbf{0}_\pm\rangle = \pm\frac{2}{L}\sin\alpha_0\langle\tilde{\Phi}_{\mathrm{I}}|\tilde{\Phi}_{\mathrm{II}}\rangle. \tag{4.20}$$

Taking the massless limit from an infinitesimally massive Ψ^a, one obtains $\alpha_0 = \frac{\pi}{2}$ [11]. The overlap of the two wavefunctions is found as

$$
\begin{aligned}
\langle\tilde{\Phi}_{\mathrm{I}}|\tilde{\Phi}_{\mathrm{II}}\rangle &= \int_0^{2\pi/gL} da\,\tilde{\Phi}_{\mathrm{I}}(a)\tilde{\Phi}_{\mathrm{II}}(a) \\
&= 2^{\frac{1}{12}}3^{\frac{2}{3}}\pi^{\frac{1}{4}}\frac{1}{\Gamma(\frac{1}{3}) + \frac{2^{4/3}\pi}{3}I}(gL)^{\frac{1}{6}}e^{-\frac{2(2\pi)^{3/2}}{3gL}}
\end{aligned} \tag{4.21}
$$

$$I = \int_0^{b_1} d\xi_1\,\xi_1^{\frac{1}{3}}\left[J_{\frac{1}{3}}(\xi_1) + J_{-\frac{1}{3}}(\xi)\right]^2. \tag{4.22}$$

The dependence of the vacuum condensate on the gauge coupling constant recalls, interestingly, the case of instanton contributions. This suggests that our result may also be derived by means of instanton calculus. The instanton-like result is a characteristic feature of this boundary condition. An interesting feature of our SUSY model compared with the non-SUSY models of [10] and [12] is the prefactor $(gL)^{1/6}$, which has a positive fractional exponent.

4.2. The (A, A) Case; AntiPeriodic Fermion and Boson

We impose the following boundary conditions on the fermions Ψ and the bosons ϕ

$$\Psi^a(x = 0) = -\Psi^a(x = L), \quad \phi^a(x = 0) = -\phi^a(x = L). \tag{4.23}$$

Obviously, the fermionic part of the Hamiltonian $H_{\rm f}$ is the same as that of the (A, P) case. As for $H_{\rm b}$, the derivation is summarized in Appendix. Similarly to (4.9), the heat kernel regularization gives the effective potential as a function of the zero mode a of the gauge fields

$$U_{M,N}(a) = \frac{2\pi}{L}\left(M - \frac{gLa}{2\pi}\right)^2 - \frac{2\pi}{L}\left(N - \frac{gLa}{2\pi}\right)^2 + V^{[N=2]}. \tag{4.24}$$

In the fundamental domain, N is given by

$$N = \begin{cases} 0, & \text{when} \quad 0 < \frac{gLa}{2} < \frac{\pi}{2}, \\ 1, & \text{when} \quad \frac{\pi}{2} < \frac{gLa}{2} < \pi. \end{cases} \tag{4.25}$$

Now let us discuss quantum mechanics for the zero mode a. The vacuum state of the (A, A) case is given by superposing the two possible states $M = 0, 1$. For $M = 1$, we obtain (see Fig. 2)

$$\tilde{H}\tilde{\Phi}_{\rm II}(a) = e\tilde{\Phi}_{\rm II}(a), \qquad \tilde{H} = -\frac{1}{2L}\frac{\partial^2}{\partial a^2} + V_{M=1}(a), \tag{4.26}$$

$$V_{M=1}(a) = \begin{cases} \infty, & \text{when} \quad a = 0, \\ U_{1,0}(a) = \frac{2}{L}(\pi - gLa) + V^{[N=2]}, & \text{when} \quad 0 < a < \pi/gL, \\ U_{1,1}(a) = V^{[N=2]}, & \text{when} \quad \pi/gL \le a < 2\pi/gL, \\ \infty, & \text{when} \quad a = 2\pi/gL. \end{cases} \tag{4.27}$$

Imposing the boundary conditions (2.20) on the wavefunction leads to the discrete energy spectrum

$$e_n = \frac{(gL)^2}{2L}n^2\left[1 - \frac{\Gamma(\frac{1}{3})}{\pi\,\Gamma(\frac{2}{3})}\left(\frac{2(gL)^2}{3}\right)^{\frac{1}{3}}\right] - \frac{1}{8L}(gL)^2, \qquad n \in \mathbf{N}. \tag{4.28}$$

Thus, the ground state $(n = 1)$ has positive energy

$$e_{\rm vac} = \frac{3}{8L}(gL)^2\left[1 - \left(\frac{2}{3}\right)^{\frac{4}{3}}\frac{2\Gamma(\frac{1}{3})}{\pi\,\Gamma(\frac{2}{3})}(gL)^{\frac{2}{3}}\right]. \tag{4.29}$$

In the $M = 0$ case, the potential $V_{M=0}(a)$ has a symmetry $V_{M=0}(a) = V_{M=1}(2\pi/gL - a)$. Hence the modified wavefunction in the $M = 0$ case $\tilde{\Phi}_{\rm I}(a)$ is given by

$$\tilde{\Phi}_{\rm I}(a) = \tilde{\Phi}_{\rm II}(2\pi/gL - a). \tag{4.30}$$

15

Note that S defined in (4.5) also provides a transformation between $\tilde{\Phi}_{\mathrm{I}}(a)$ and $\tilde{\Phi}_{\mathrm{II}}(a)$.

From these results we are able to write down the full vacuum state vector, which we take to be an eigenstate of the symmetry operator S with eigenvalue ± 1:

$$|\mathbf{0}_\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|\tilde{\Phi}_{\mathrm{I}}(a)\rangle|0_\varphi; M=0\rangle \pm e^{i\alpha_0}|\tilde{\Phi}_{\mathrm{II}}(a)\rangle|0_\varphi; M=1\rangle\right)|0_\Psi\rangle|0_\xi\rangle|0_\phi\rangle, \tag{4.31}$$

where we assume that $|0_\Psi\rangle|0_\xi\rangle|0_\phi\rangle$ is invariant under S.

We find the condensate on this vacuum state $|\mathbf{0}_\pm\rangle$ as

$$\langle\mathbf{0}_\pm|\bar{\Psi}^a\Psi^a|\mathbf{0}_\pm\rangle = \pm\frac{2}{L}\sin\alpha_0\langle\tilde{\Phi}_{\mathrm{I}}|\tilde{\Phi}_{\mathrm{II}}\rangle = \pm\frac{3\sqrt{2}}{4\pi^2 L}\left(\Gamma(1/3)\right)^3\sin\alpha_0\,(gL)^2. \tag{4.32}$$

As in the (A, P) case, we take $\alpha_0 = \frac{\pi}{2}$.

4.3. The (P, A) Case; Periodic Fermion and AntiPeriodic Boson

We impose the following boundary condition on the fermions Ψ and bosons ϕ

$$\Psi(x=0) = \Psi(x=L), \quad \phi(x=0) = -\phi(x=L). \tag{4.33}$$

Similarly to other cases, the effective potential for the zero modes a is found to be

$$U_{M,N}(a) = \frac{2\pi}{L}\left(M - \frac{1}{2} - \frac{gLa}{2\pi}\right)^2 - \frac{2\pi}{L}\left(N - \frac{gLa}{2\pi}\right)^2 + \frac{\pi}{4L} + V^{[N=2]}. \tag{4.34}$$

The integer N is given by (4.25). To minimize the vacuum energy for fixed a, we have $M = 1$. The effective potential (Fig. 3) of the (P, A) case is given by

$$V(a) = \begin{cases} \infty, & \text{when } a = 0, \\ U_{1,0}(a) = -ga + \frac{3\pi}{4L} + V^{[N=2]}, & \text{when } 0 < a < \pi/gL, \\ U_{1,1}(a) = ga - \frac{5\pi}{4L} + V^{[N=2]}, & \text{when } \pi/gL \le a < 2\pi/gL, \\ \infty, & \text{when } a = 2\pi/gL. \end{cases} \tag{4.35}$$

The Schrödinger equation takes the form

$$\tilde{H}\tilde{\Phi}(a) = e\tilde{\Phi}(a), \qquad \tilde{H} = -\frac{1}{2L}\frac{\partial^2}{\partial a^2} + V(a). \tag{4.36}$$

Using $gL \ll 1$, we obtain the eigenvalues of the Hamiltonian

$$e_n = -\frac{\pi}{4L} + \frac{1}{2L}\left(3a_n gL\right)^{\frac{2}{3}} - \frac{1}{8L}(gL)^2, \qquad n \in \mathbf{N}, \tag{4.37}$$

16

where the a_n are defined using $\xi_2(a)$ in eq.(A.15) of appendix

$$I_{-\frac{1}{3}}\left(\xi_2(a=0)\right) J_{-\frac{2}{3}}(a_n) - I_{\frac{1}{3}}\left(\xi_2(a=0)\right) J_{\frac{2}{3}}(a_n) = 0, \qquad a_n > 0. \tag{4.38}$$

Thus the ground state $(n = 1)$ has nonvanishing energy.

We find that the vacuum state of the (P, A) case can be written as

$$\begin{aligned}
|\mathbf{0}_\Omega\rangle &\equiv |\tilde{\Phi}(a)\rangle |0_\varphi; M = 1\rangle |0_\Psi\rangle |0_\xi\rangle |0_\phi\rangle |\Omega\rangle, \\
|\mathbf{0}_{\tilde{\Omega}}\rangle &\equiv |\tilde{\Phi}(a)\rangle |0_\varphi; M = 1\rangle |0_\Psi\rangle |0_\xi\rangle |0_\phi\rangle |\tilde{\Omega}\rangle.
\end{aligned} \tag{4.39}$$

We find the vacuum condensate $\left|\langle\mathbf{0}|\bar{\Psi}^a\Psi^a|\mathbf{0}\rangle\right| = \frac{1}{L}$ for both $|\mathbf{0}\rangle = |\tilde{\Phi}(a)\rangle$ and $|\mathbf{0}_{\tilde{\Omega}}\rangle$.

5. Summary

This paper discusses two-dimensional supersymmetric Yang-Mills theories (SYM_2), which are defined on the compactified spatial region with interval L. It is possible to gauge away all gauge fields except for the zero modes. Under the condition $gL \ll 1$, the vacuum structures of SYM_2 are discussed by solving the quantum mechanics of the zero modes. The Jacobian associated with the change of variable to zero modes gives rise to nontrivial results. The vacuum states are described in terms of the wavefunctions depending on the zero modes.

Depending on the choice of boundary conditions, there are four different cases. The first is the (P, P) case. The ground states of this case turn out to have vanishing energy; however we cannot count the zero energy states because of the zero modes of the scalar field. Such difficulties also appear in [4]. The gaugino bilinear condensate is calculated in the (P, P) case. It is found that the gaugino condensate is independent of the gauge coupling constant g.

The paper also discusses three other cases having different boundary conditions for the spinor fields $\Psi(x)$ and/or the scalar fields $\phi(x)$. The ground states of these cases commonly possess nonvanishing energy. This suggests that these three boundary conditions do not preserve supersymmetry. Their vacuum structures are quite different from each other. For example, the gaugino condensates depend on the coupling constant g if and only if the boundary conditions of $\Psi(x)$ are antiperiodic.

Among the four cases, the one of great interest is the (A, P) case. The vacuum condensate includes nontrivial structures, which resemble those related to the contribution of the instantons. This similarity indicates that our results may also be obtained by using Smilga's approach [10]. In fact, the boundary conditions of the (A, P) case are very similar to those of Euclidean field theories with finite imaginary time. Moreover, the potential energy of the (A, P) case exhibits a double-well structure, which has already been extensively studied from the viewpoint of instantons [19].

It is worth comparing the discussion of the Witten index in 2-dimentions [4] with that in 4-dimentions [15]. We notice two significant differences: First, the spectrum in the 2-dimentional case is continuous owing to the zero modes of the scalar field. Therefore the Witten index is ill-defined. It is necessary to put a cut-off for the space of the zero modes in order to make the Witten index well-defined. On the other hand, there are no zero modes of the scalar field in the 4-dimentional case. Therefore the spectrum is discrete, and the Witten index is well-defined. Second, the 4-dimentional theory has a complex Weyl spinor while the 2-dimensional case contains a Majorana spinor. In the 4-dimensional case, the complex Weyl spinor can be written in the form of the creation and annihilation operators $a_\alpha^{\dagger\sigma}$ and a_α^σ ($\alpha = 1, 2; \sigma = 1, \ldots, r$) satisfying $\{a_\alpha^\sigma, a_\beta^{\dagger\tau}\} = \delta_{\alpha\beta}\delta^{\sigma\tau}$, where $r = N - 1$. Let $|\Omega\rangle$ be the Clifford vacuum which is annihilated by a_α^σ. Depending on whether $|\Omega\rangle$ is invariant or pseudo-invariant state under the Weyl group, there are two possible cases for the Witten index. When $|\Omega\rangle$ is invariant, the Weyl invariant zero energy states can be given in the form of $|\Omega\rangle, U|\Omega\rangle, \ldots, U^r|\Omega\rangle$, where U is the Weyl invariant operator given by $U = a_\alpha^{\dagger\sigma} a_\beta^{\dagger\sigma} \epsilon^{\alpha\beta}$. It then follows $\mathrm{tr}(-1)^F = r + 1$. In the case where $|\Omega\rangle$ is pseudo-invariant, the Weyl invariant states can be given by acting on $|\Omega\rangle$ the pseudo-invariant operators $V_{\alpha_1\cdots\alpha_r} = a_{\alpha_1}^{\dagger\sigma_1}\cdots a_{\alpha_r}^{\dagger\sigma_r}\epsilon_{\sigma_1\cdots\sigma_r}$, which have spin $r/2$. Thus, one obtains $\mathrm{tr}(-1)^F = (-1)^{r+1}(r + 1)$. These two results imply that there is an ambiguity of the sign of the Witten index. In four dimensions, this ambiguity cannot be removed. On the other hand, in the 2-dimensional case, there is only one ground state and moreover such an ambiguity of the sign does not appear. In the present case, the zero modes of the Majorana spinor satisfy $\{\lambda^\sigma, \lambda^{\dagger\tau}\} = \delta^{\sigma\tau}$ (see (3.31)). If the Clifford vacuum is Weyl-invariant, this is the unique zero energy state because there is no Weyl-invariant operator. If the Clifford vacuum is pseudo-invariant, the allowed zero energy state is given by acting the pseudo-invariant operator $V = \epsilon_{\sigma_1\cdots\sigma_r}\lambda^{\dagger\sigma_1}\cdots\lambda^{\dagger\sigma_r}$. Thus, in both cases, there is only one ground state as long as the subtleties associated with the zero modes of the scalar field are overcome. One can always redefine the fermionic number to change fermions into bosons and *vice versa*, since the fermionic number does not have an intrinsic meaning in two dimensions. We have explicitly demonstrated the above mechanism in the case of $SU(2)$.

Appendix

Here we discuss the vacuum states of H_{b} in the (A, A), and the (P, A) cases, and derive the vacuum energy of these cases. First we consider the (A, A) case. The mode expansions of the

scalar fields ξ, η, ϕ^3, and π^3 take the form

$$\xi(x) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2LE_k}} \left(e_k + f_k^\dagger \right) e^{i2\pi(k+1/2)x/L}, \quad E_k = \left| \frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right|, \quad \text{(A.1)}$$

$$\eta(x) = \sum_{k=-\infty}^{\infty} i\sqrt{\frac{E_k}{2L}} \left(-e_k + f_k^\dagger \right) e^{i2\pi(k+1/2)x/L}, \quad \text{(A.2)}$$

$$\phi^3(x) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2LF_k}} \left(g_k + g_{-k-1}^\dagger \right) e^{i2\pi(k+1/2)x/L}, \quad F_k = \left| \frac{2\pi \left(k + \frac{1}{2} \right)}{L} \right|, \quad \text{(A.3)}$$

$$\pi^3(x) = \sum_{k=-\infty}^{\infty} i\sqrt{\frac{F_k}{2L}} \left(-g_k + g_{-k-1}^\dagger \right) e^{i2\pi(k+1/2)x/L}, \quad \text{(A.4)}$$

where e_k, f_k and g_k satisfy the commutation relations,

$$\left[e_k, e_{k'}^\dagger \right] = \left[f_k, f_{k'}^\dagger \right] = \left[g_k, g_{k'}^\dagger \right] = \delta_{k,k'}. \quad \text{(A.5)}$$

We evaluate the Hamiltonian $H_{\text{b,off}}$ in (3.5) by defining an integer $N = \left[\frac{gaL}{2\pi} + \frac{1}{2} \right]$

$$H_{\text{b,off}} = \sum_{k=-\infty}^{\infty} E_k \left(e_k^\dagger e_k + f_k^\dagger f_k \right)$$
$$- \sum_{k=-\infty}^{N-1} \left(\frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right) + \sum_{k=N}^{\infty} \left(\frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right). \quad \text{(A.6)}$$

Then the vacuum state satisfying $e_k|0_\xi\rangle = f_k|0_\xi\rangle = 0$ has energy given by

$$H_{\text{b,off}}|0_\xi\rangle = \left[-\sum_{k=-\infty}^{N-1} \left(\frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right) + \sum_{k=N}^{\infty} \left(\frac{2\pi \left(k + \frac{1}{2} \right)}{L} - ga \right) \right] |0_\xi\rangle$$
$$\equiv V_{\text{b,off}}(a)|0_\xi\rangle. \quad \text{(A.7)}$$

We also obtain the Hamiltonian $H_{\text{b,diag}}$ in (3.4) as

$$H_{\text{b,diag}} = \sum_{k\geq 0} \frac{2\pi \left(k + \frac{1}{2} \right)}{L} \left(g_k^\dagger g_k + g_{-k-1}^\dagger g_{-k-1} + 1 \right). \quad \text{(A.8)}$$

Defining the vacuum state as $g_k|0_\phi\rangle = 0$ for all k, the vacuum energy is given by

$$V_{\text{b,diag}} = \sum_{k\geq 0} \frac{2\pi \left(k + \frac{1}{2} \right)}{L}. \quad \text{(A.9)}$$

We find for the vacuum energy of the antiperiodic boson

$$H_{\mathrm{b}}|0_\xi\rangle|0_\phi\rangle = \left(V_{\mathrm{b,off}}(a) + V_{\mathrm{b,diag}}\right)|0_\xi\rangle|0_\phi\rangle. \tag{A.10}$$

Solving the Schrödinger equation (4.26), we obtain the eigenfunctions

$$\tilde{\Phi}_{\mathrm{II}}(a) = \begin{cases} \xi_2^{\frac{1}{3}}(a)\left\{-A_+ I_{\frac{1}{3}}\left(\xi_2(a)\right) + A_- I_{-\frac{1}{3}}\left(\xi_2(a)\right)\right\}, & \text{when } 0 \le a \le s, \\ \xi_1^{\frac{1}{3}}(a)\left\{A_+ J_{\frac{1}{3}}\left(\xi_1(a)\right) + A_- J_{-\frac{1}{3}}\left(\xi_1(a)\right)\right\}, & \text{when } s < a \le \pi/gL, \\ B\sin\left(\sqrt{2L(e-V^{[N=2]})}\left(\frac{2\pi}{gL} - a\right)\right), & \text{when } \pi/gL < a \le 2\pi/gL, \end{cases} \tag{A.11}$$

$$\xi_1(a) \equiv \frac{4}{3}\sqrt{gL}\,(a-s)^{\frac{3}{2}}, \quad \xi_2(a) \equiv \frac{4}{3}\sqrt{gL}\,(-a+s)^{\frac{3}{2}}, \quad s = \frac{\pi}{gL} - \frac{1}{2g}\left(e - V^{[N=2]}\right). \tag{A.12}$$

Energy eigenvalue condition is given by

$$-\cot\left(\frac{\pi}{gL}\sqrt{2L(e-V^{[N=2]})}\right) \tag{A.13}$$

$$= \frac{I_{-\frac{1}{3}}\left(\xi_2(a=0)\right)J_{-\frac{2}{3}}\left(\xi_1(a=\frac{\pi}{gL})\right) - I_{\frac{1}{3}}\left(\xi_2(a=0)\right)J_{\frac{2}{3}}\left(\xi_1(a=\frac{\pi}{gL})\right)}{I_{-\frac{1}{3}}\left(\xi_2(a=0)\right)J_{\frac{1}{3}}\left(\xi_1(a=\frac{\pi}{gL})\right) + I_{\frac{1}{3}}\left(\xi_2(a=0)\right)J_{-\frac{1}{3}}\left(\xi_1(a=\frac{\pi}{gL})\right)}$$

For the (P, A) case, the solutions to the Schrödinger equation (4.36) are given as

$$\tilde{\Phi}(a) = \begin{cases} \xi_2^{\frac{1}{3}}(a)\left\{-A_+ I_{\frac{1}{3}}\left(\xi_2(a)\right) + A_- I_{-\frac{1}{3}}\left(\xi_2(a)\right)\right\}, & \text{when } 0 \le a \le s, \\ \xi_1^{\frac{1}{3}}(a)\left\{A_+ J_{\frac{1}{3}}\left(\xi_1(a)\right) + A_- J_{-\frac{1}{3}}\left(\xi_1(a)\right)\right\}, & \text{when } s < a \le \pi/gL, \\ \xi_1^{\frac{1}{3}}(a')\left\{A_+ J_{\frac{1}{3}}\left(\xi_1(a')\right) + A_- J_{-\frac{1}{3}}\left(\xi_1(a')\right)\right\}, & \text{when } \pi/gL < a \le 2\pi/gL - s, \\ \xi_2^{\frac{1}{3}}(a')\left\{-A_+ I_{\frac{1}{3}}\left(\xi_2(a')\right) + A_- I_{-\frac{1}{3}}\left(\xi_2(a')\right)\right\}, & \text{when } 2\pi/gL - s < a \le 2\pi/gL, \end{cases} \tag{A.14}$$

$$\xi_1(a) = \frac{2}{3}\sqrt{2gL}\,(a-s)^{\frac{3}{2}}, \quad \xi_2(a) = \frac{2}{3}\sqrt{2gL}\,(-a+s)^{\frac{3}{2}}, \tag{A.15}$$

$$a' = \frac{2\pi}{gL} - a, \qquad s = -\frac{1}{g}\left(e - \frac{3\pi}{4L} - V^{[N=2]}\right). \tag{A.16}$$

References

[1] N. Seiberg and E. Witten, *Nucl. Phys.* **B431** (1994) 484.

[2] N. Seiberg, *Nucl. Phys.* **B435** (1995) 129; D. Kutasov, *Phys. Lett.* **B351** (1995) 230.

[3] G. 't Hooft, *Nucl. Phys.* **B72** (1974) 461.

[4] M. Li, *Nucl. Phys.* **B446** (1995) 16.

[5] Y. Matsumura, N. Sakai and T. Sakai, *Phys. Rev.* **D52** (1995) 2446.

[6] S. Dalley and I. R. Klebanov, *Phys. Rev.* **D47** (1993) 2517; G. Bhanot, K. Demeterifi and I. R. Klebanov, *Phys. Rev.* **D48** (1993) 4980; K. Demeterifi, I. R. Klebanov and G. Bhanot, *Nucl. Phys.* **B418** (1994) 15.

[7] D. Kutasov, *Nucl. Phys.* **B414** (1994) 33; J. Boorstein and D. Kutasov, *Nucl. Phys.* **B421** (1994) 263.

[8] D. J. Gross, I. R. Klebanov, A. V. Matytsin and A. V. Smilga, *Nucl. Phys.* **B461** (1996) 109.

[9] I. I. Kogan, *Phys. Rev.* **D49** (1994) 6799.

[10] A. V. Smilga, *Phys. Rev.* **D49** (1994) 6836.

[11] F. Lenz, H. W. L. Naus, and M. Thies, *Ann. Phys.* (N.Y.) **233** (1994) 317.

[12] F. Lenz, M. Shifman, and M. Thies, *Phys. Rev.* **D51** (1995) 7060.

[13] H. C. Pauli, A. C. Kalloniatis and S. S. Pinsky, *Phys. Rev.* **D52** (1995) 1176; A. C. Kalloniatis, hep-th.9509027; S. S. Pinsky and A. C. Kalloniatis, *Phys. Lett.* **B365** (1996) 225.

[14] I. I. Kogan and A. R. Zhitnitsky, *Nucl. Phys.* **B465** (1996) 99.

[15] E. Witten, *Nucl. Phys,* **B202** (1982) 253.

[16] A. V. Smilga, *Sov. Phys. JETP* **64** (1986) 8; B. Yu. Blok and A. V. Smilga *Nucl. Phys.* **B287** (1987) 589; A. V. Smilga *Nucl. Phys.* **B291** (1987) 241.

[17] T. Maskawa and K. Yamawaki, *Prog. Theor. Phys.* **56** (1976) 270; C. Bender, S. Pinsky and B. van de Sanda, *Phys. Rev.* **D48** (1993) 816; A.C. Kalloniatis and D.G. Robertson, *Phys. Rev.* **D50** (1994) 5262.

[18] S. Kojima, N. Sakai and T. Sakai, *Prog. Theor. Phys.* **95** (1996) 621.

[19] A. M. Polyakov, *Nucl. Phys.* **B120** (1977) 429.

[20] S. Ferrara, *Lett. Nouvo Cimen.* **13** (1965) 629.

Figure captions

Fig. 1 Effective potential for the (A, P) case with $M = 1$.

Fig. 2 Effective potential for the (A, A) case with $M = 1$.

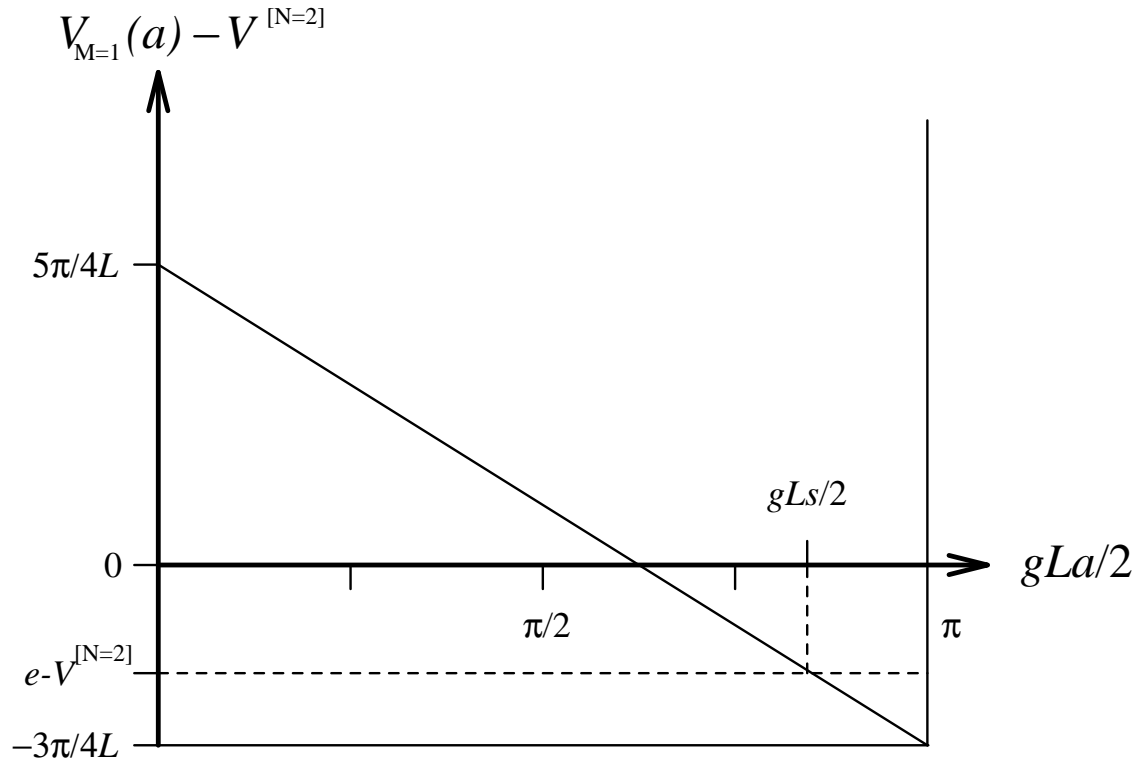Fig. 3 Effective potential for the (P, A) case.

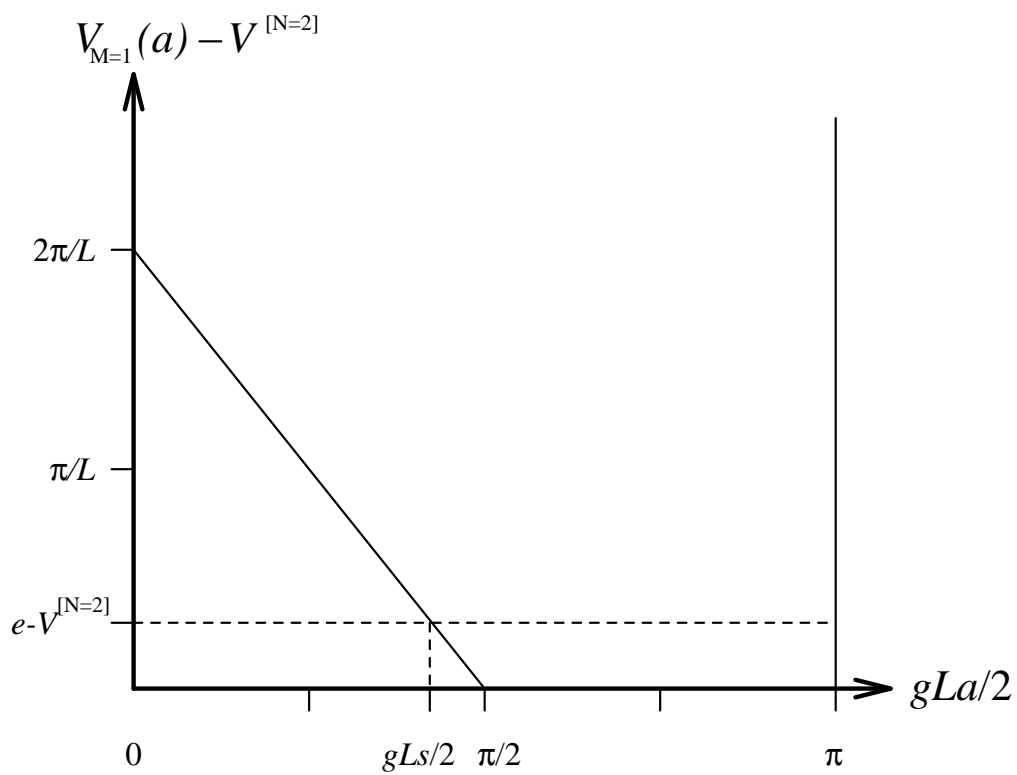Figure 1: Effective potential for the (A, P) case with $M = 1$.

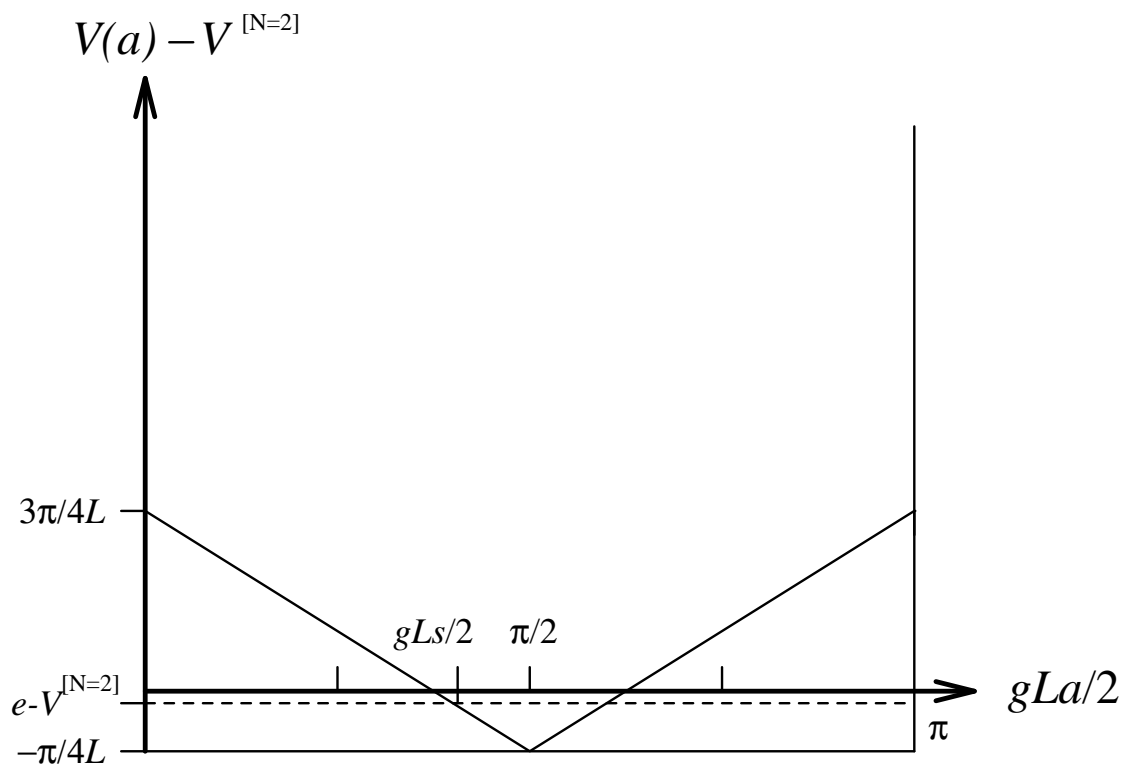Figure 2:   Effective potential for the (A, A) case with $M = 1$.

Figure 3: Effective potential for the (P, A) case.